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THE TAUB-NUT CASE

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HYPERKAHLER METRICS BUILDING IN THE 1+3 REPRESENTATION. THE TAUB-NUT CASE *

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ABSTRACT

We construct the HST type hyperkahler N=2 supersymmetric non-linear σ-model. We show that for 4n dimensional hyperkahler manifolds, one needs in general $\frac{n(n+1)}{2}$ Kahler-like potentials. The n=1 and n=2 cases are studied in detail. The SU(2) x U(1) Taub-Nut model in the HST hypermultiplet representation is discussed as an illustrating example. The corresponding Taub-Nut metric is worked out explicitly. Some general features of these models are discussed.

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1. INTRODUCTION

The construction of hyperkahler metrics are of interest from both the mathematical and physical points of view. The hyperkahler manifolds form a subset of Ricci flat manifolds for which the complex structure has been generalized to a quaternionic one [1],[2]. The necessary and sufficient conditions for the existence of such manifolds were conjectured by Calaby [3] and proved by Yau [4]. Moreover, hyperkahler manifolds correspond to finite supersymmetric non-linear σ-models [1],[5],[6] and hence, besides their relationship with the vacuum Einstein equation [7], are exact solutions to the superstring equations of motion at this tree level [8]. However, a systematic way of constructing explicit hyperkahler metrics is still an unsolved question.

The introduction of the harmonic superspace techniques by Galperin et al. [9],[10] offers a powerful method of constructing hyperkahler metrics by solving the auxiliary field equations of motion [11],[12],[13],[14]. These hyperkahler manifolds are 4n, n = integer, dimensional manifolds which can be parametrized in terms of different sets of coordinates. For the n=1 case for instance, the manifold may be parametrized either by four real coordinates of by two complex variables [1]. For hyperkahler metrics which admit an SU(2) isometry group, it is interesting to use a manifestly SU(2) covariant coordinate. In the complex parametrization, the two complex coordinates are interpreted as an SU(2) isospinor while in the real coordinate system, the four coordinates are put in a isosinglet and isotriplet. These two coordinate systems are dual to each other [9],[15],[16] and allows to write down covariant SU(2) quantities such as the explicit hyperkahler metrics with SU(2) isometry group.

In this paper we construct a class of N=2 supersymmetric non-linear σ-models in the 1+3 representation preserving the SU(2) automorphism group. We show that for 4n hyperkahler manifolds, one needs in general n(n+1)/2 Kahler-like potentials and present a method of constructing explicit hyperkahler metrics having an SU(2)xG isometry group. The procedure is illustrated on the SU(2)xU(1) Taub-Nut metric in the 1+3 coordinate system.

The presentation is as follows:

In Sec. 2, we construct the more general class of N=2 supersymmetric non-linear σ-models preserving the automorphism group in the 1+3 representation. We first examine the case of a non-renormalizable self-interacting real HST

hypermultiplet. We discuss the constraints equations which express the auxiliary fields in terms of the N=2 on shell degrees of freedom. The bosonic part of the model is studied in detail together with the component field Lagrangian. A relation with N=1 Kahler geometry is pointed out. Then, we give the generalization of the model for more than one HST hypermultiplet above all models with SO(n) symmetry. The $SO(2)\approx U(1)$ case is formulated explicitly.

In Sec. 3, we illustrate the previous study by discussing the case of SU(2)xU(1) Taub-Nut metric. We show how the model can be derived from the requirement of the SU(2)xU(1) symmetry. Then we give in detail the solutions of the constraint equations and work out the corresponding SU(2)xU(1) Taub-Nut hyperkahler metric in the (1+3) representations. Some special features of the model are pointed out.

Sec. 4 is devoted to discussions and conclusion.

2. Ω TYPE SELF-COUPLINGS AND N=2 SUPERSYMMETRIC NON-LINEAR σ-MODELS

We start by recalling the free action of a real HST hypermultiplet Ω in the harmonic superspace

$$\begin{aligned} S_0 &= \int d^4x_A\, du\, \mathcal{L}_0 \\ \mathcal{L}_0 &= \int d^4\theta^+ (D^{++}\Omega)^2 \quad , \end{aligned} \tag{2.1}$$

where $D^{++} = \partial^{++} - 2i\theta^+\sigma^\mu\bar\theta^+\partial_\mu$ is the covariant harmonic derivation and Ω the HST analytic superfield given by:

$$\begin{aligned}\Omega = \omega + \sqrt{2}\,\theta^+\zeta^- + \sqrt{2}\,\bar\theta^+\bar\zeta^- + \theta^{+2}F^{--} + \bar\theta^{+2}\bar F^{--} + i\theta^+\sigma^\mu\bar\theta^+A_\mu^{--} \\ + \sqrt{2}\,\bar\theta^{+2}\theta^+\xi^{(---)} + \sqrt{2}\,\theta^{+2}\bar\theta^+\bar\xi^{(---)} + \theta^{+2}\bar\theta^{+2}\Delta^{(----)} .\end{aligned} \tag{2.2}$$

Each component $\varphi^{(q)}$ of the θ^+ expansion (2.2), may be written in a harmonic series as:

$$\varphi^{(q)}(x_A,U) = \sum_{m\geq 0,\, q\geq 0} \varphi^{(i_1\cdots i_{n+q}, j_1\cdots j_m)}(x_A)\, U^+_{i_1}\cdots U^+_{i_{n+q}} U^-_{j_1}\cdots U^-_{j_m} . \tag{2.3}$$

The introduction of all this tail of auxiliary fields is the price one pays to realize N=2 supersymmetry off-shell.

The equation of motion reads as:

$$D^{++2}\Omega = 0 \quad . \tag{2.4}$$

Equivalently in terms of the component fields of Eq.(2.2), we have:

$$\begin{cases} \partial^{++2}\omega = 0 \\ \partial^{++2}\zeta^- = 0 \end{cases} \tag{2.5}$$

$$\begin{cases} \partial^{++2}F^{--} = 0 \\ \partial^{++2}A_\mu^{--} = 4\,\partial^{++}\partial_\mu\omega \end{cases} \tag{2.6}$$

and an extra set of equations carrying the space-time evolution which are kept off-shell. The solutions of the constraint equations (2.5) are given by the N=2 on-shell degrees of freedom $(0^4, 1/2^2)$ namely:

$$\begin{aligned} \omega(x_A,U) &\equiv f(x_A,U) = f^0(x_A) + f^{(ij)}(x_A)\, U^+_i U^-_j \\ \zeta^-(x_A,U) &\equiv \Psi^-(x_A,U) = \Psi^i(x_A)\, U^-_i \end{aligned} \tag{2.7}$$

In what follows, we shall disregard the spinor fields and deal with the bosonic fields only.

The generalization of the action (2.1) in terms of one real HST hypermultiplet to N=2 sypersymmetric non-linear σ-models preserving the SU(2) automorphism group is unique and reads as:

$$\mathcal{L}_\sigma = P(\lambda,\Omega)\, (D^{++}\Omega)^2 \quad , \tag{2.8}$$

$P(\lambda,\Omega)$ is an arbitrary dimensionless function depending on Ω and the dimensionfull coupling parameter λ. It has in general the form

$$P(\lambda,\Omega) = 1 + \lambda . R(\Omega,\lambda) \tag{2.9}$$

The real superfield function $P(\lambda,\Omega)$ is comparable in some sense to the Kahler potential of the N=1 supersymmetric theories [17],[18] at least in absence of internal symmetries. When the latter is present, the weight $P(\lambda,\Omega)$ of Eq.(2.8) becomes a tensor of rank two (see later on).

The equation of motion of the Lagrangian (2.8) is given by:

$$D^{++2}\Omega + \frac{P_{,\Omega}}{2P}\left(D^{++}\Omega\right)^2 = 0 \qquad (2.10)$$

where $P_{,\Omega} = \partial P/\partial\Omega$ and the arguments of P are dropped for simplicity.

Eq. (2.10) is a non-linear harmonic differential equation of second order with non-constant coefficients and therefore not easy to solve. This superfield equation splits into three different sets according to the canonical dimension c.d. of the component fields of the θ^+ expansion of Eq.(2.2). The first set $(\xi^{(---)}, \Delta^{(---)})$ consists of the auxiliary fields of c•d 5/2 and 3 respectively and appear therefore as Lagrange parameters leading to constraint equations on the harmonic series type (2.3). These constraints are obtained by projecting Eq.(2.10) along the $\theta^+ = \bar\theta^+ = 0$ and $\theta^+,\bar\theta^+$. One gets for the pure bosonic equation:

$$\partial^{++2}\omega + \frac{P_{,\omega}}{2P}\left(\partial^{++}\omega\right)^2 = 0 \qquad (2.11)$$

The second set (F^{--}, A_μ^{--}) consists of auxiliary fields of c•d equal to two and come quadratically in the component field action. They contribute in general to the potential of the model once they are eliminated through their equations of motion [11],[13]. They are given by:

$$\partial^{++2}F^{--} + \left[\frac{P_{,\omega}}{P}\partial^{++}\omega\right]\partial^{++}F^{--} + \left[\frac{2P_{,\omega}\partial^{++2}\omega + P_{,\omega\omega}\left(\partial^{++}\omega\right)^2}{2P}\right]F^{--} = 0 \qquad (2.12)$$

$$\partial^{++2}A_\mu^{--} + \left[\frac{P_{,\omega}}{P}\partial^{++}\omega\right]\partial^{++}A_\mu^{--} + \left[\frac{2P_{,\omega}\partial^{++2}\omega + P_{,\omega\omega}\left(\partial^{++}\omega\right)^2}{2P}\right]A_\mu^{--}$$
$$-4\left[\partial^{++}\partial_\mu\omega + \frac{P_{,\omega}\partial^{++}\omega}{2P}\partial_\mu\omega\right] = 0 \qquad (2.13)$$

The last set (ω, ζ^-) contains the propagating degrees of freedom and will be kept off shell.

Now, we are in position to discuss the solutions of the constraint equations (2.11)-(2.13). The first (2.11), as well as the two others, is a differential equation on the S^2 sphere. It expresses the ω dependence on the harmonic variables $u_i^\pm$. This dependence is of course correlated to the form of the interaction carried by the coefficient $P_{,\omega}/2p$. Moreover, because of N=2 supersymmetry which constraints the dimensionality of the N=2 supersymmetric non-linear σ-models (2.8) to be four, we choose the solution of Eq.(2.11) of the form:

$$\omega = \omega(f, \lambda) \qquad (2.14)$$

$f = f^0 + f^{(ij)}u_i^+u_j^-$ is the free limit of ω and carries the four real scalar degrees of freedom of the N=2 on-shell multiplet as mentioned earlier (Eq.(2.7)). The choice (2.14) manifests itself through a consistency condition on the scalar potential $P(\lambda,\Omega)$. Indeed, putting Eq.(2.14) in Eq.(2.11), one gets:

$$\partial^{++2}f + \left[\frac{\omega_{,ff}}{\omega_{,f}} + \frac{P_{,f}}{2P}\right]\left(\partial^{++}f\right)^2 = 0 \qquad (2.15)$$

$$\omega_{,ff} = \frac{\partial^2\omega}{\partial f^2}, \qquad \omega_{,f} = \frac{\partial\omega}{\partial f}$$

$$P_{,f} = \frac{\partial P}{\partial\omega}\cdot\frac{\partial\omega}{\partial f} = P_{,\omega}\cdot\omega_{,f} \qquad (2.16)$$

Using the fact that $\partial^{++2}f = 0$ and $\partial^{++}f = f^{++} \neq 0$, Eq.(2.15) reduces to:

$$\frac{d}{df}\ln\left(\sqrt{P}\cdot\omega_{,f}\right) = 0 \qquad (2.17)$$

and thus

$$P = \mathrm{const}\left(\frac{d\omega}{df}\right)^{-2}. \qquad (2.18)$$

In the limit of a free theory ($\lambda=0$), the constant is shown to be equal to unity. Therefore, we have the following relation:

$$P(f)\cdot\omega_{,f}^2 = 1. \qquad (2.19)$$

The equation (2.12) which is more difficult than the previous one, has a trivial solution for $F^{--} = 0$. However, since the study of the hypermultiplet theory in terms of the Fayet-Sohnius (F-S) multiplet has shown that the bosonic part of the F-terms are proportional to the central charge eigenvalues [13] and knowing that the <u>real</u> HST hypermultiplet cannot have a central charge [16], one may expect the solution of Eq.(2.12) to be $F^{--} = 0$. We are not going to prove it and thus we shall keep F^{--} in the final form of the Lagrangian.

The last equation (2.13) giving the auxiliary field A_μ^{--} is also a harmonic differential equation of second order depending on the scalar weight and its first and second derivatives $P_{,\omega}$ and $P_{,\omega\omega}$. The general form of the solution reads as:

$$A_\mu^{--} = \gamma(f,\lambda)\,\partial_\mu f^{--} + \delta^{--}(f,\lambda)\,\partial_\mu f + \eta^{(-4)}(f,\lambda)\,\partial_\mu f^{++} \tag{2.20}$$

$\gamma(f,\lambda)$, $\delta^{--}(f,\lambda)$ and $\eta^{(-4)}(f,\lambda)$ are functions to be determined.

For simplicity, we shall use A_μ^{--} in the final form of the Lagrangian but keeping in mind that A_μ^{--} may be completely expressed in terms of f and $\partial_\mu f$.

Integrating with respect to θ^+ and using the constraint equations (2.11)-(2.13), the N=2 supersymmetric non-linear σ-model may be expressed in terms of the component fields ω, F^{--} and A_μ^{--} as follows:

$$\mathcal{L}_\sigma = -\frac{1}{2}\Big\{ +4\,P(\omega)\,(\partial_\mu\omega)^2 + 2\,P_{,\omega}\,\partial^{++}\omega\, A_\mu^{--}\partial^\mu\omega + 4\,P(\omega)\,A_\mu^{--}\partial^{++}\partial^\mu\omega + \left[P_{,\omega\omega}\,(\partial^{++}\omega)^2\, F^{--}\bar{F}^{--}\right] \Big\}. \tag{2.21}$$

Or equivalently in terms of the N=2 on shell degree of freedom f^0 and $f^{(ij)}$:

$$\mathcal{L}_\sigma(f) = \frac{1}{2}\Big\{-4\,(\partial_\mu f)^2 + 4\,P(f)\,\omega_{,f}\,A_\mu^{--}\partial^\mu f^{++} + \left[P_{,ff} + \frac{1}{2P}(P_{,f})^2\right] f^{++2}\, F^{--}\bar{F}^{--}\Big\} \tag{2.22}$$

Notice that switching off the interaction in Eq.(2.22): $P_{,\omega} = 0$, one recovers the free theory. Moreover, the term proportional to A_μ^{--} together with the free term give the metric of the N=2 non-linear σ-models (2.8). This metric is related to the scalar weight $P(\Omega,\lambda)$ and should exhibit a hyperkahler structure [5],[1],[18]. The last term, which is proportional to the F^{--} term, gives the scalar potential of the model.

The next step is to perform the integration with respect to the harmonic variables. This depends however on the explicit form of the scalar weight $P(\lambda,\Omega)$ and the solutions of Eqs.(2.12) and (2.13). Nevertheless, one may easily see that the general form of the N=2 supersymmetric non-linear σ-models (2.8), once the integrations with respect to both θ^+'s and the harmonic variables are performed, leads to:

$$\mathcal{L}_\sigma = A(f)\,(\partial_\mu f^0)^2 + B_{ij}(f)\,\partial_\mu f^0\,\partial^\mu f^{(ij)} + C_{ijkl}(f)\,\partial_\mu f^{(ij)}\,\partial^\mu f^{(kl)}. \tag{2.23}$$

$A(f)$, $B_{ij}(f) = B_{ji}(f)$ and $C_{ijkl} = C_{jilk} = C_{klij}$ are metric tensors which could be specified explicitly for a given $P(\lambda,\Omega)$.

Notice also that $B_{ij}(f)$ is proportional to the coupling parameter λ and in the limit of a free theory, one has :

$$\begin{aligned} A &= const \\ B_{ij} &= 0 \\ C_{ijkl} &= const\ \varepsilon_{i(k}\,\varepsilon_{l)j}\,. \end{aligned} \tag{2.24}$$

We shall discuss the interesting case of the Taub-Nut metric in more detail later on. For the time being, let us examine the generalization of these types of N=2 non-linear σ-models to the case of more than one real HST hypermultiplet. The introduction of more than one HST hypermultiplet allows us to gain an extra symmetry which helps to solve the non-linear differential constraint equations type (2.11)-(2.13).

Consider a collection of HST hypermultiplet Ω^a; $a = 1,\dots,n$ transforming under an internal symmetry group representation, say the vector representation of $SO(n)$. The N=2 non-linear σ-model (2.8) may be generalized as follows:

$$\mathcal{L}_\sigma = P_{ab}(\lambda,\Omega)\, D^{++}\Omega^a\, D^{++}\Omega^b\,. \tag{2.25}$$

where $P_{ab}(\lambda,\Omega) = P_{ba}(\lambda,\Omega)$ is a symmetric tensor of $\frac{n(n+1)}{2}$ different components. Eq.(2.25) is the most general quantity preserving the SU(2) automorphism group rotating the two supersymmetries. The previous discussion can be easily extended to the model (2.25) and one is finally left with the following component field N=2 supersymmetric Lagrangian:

$$\mathcal{L}_\sigma = A^{ab}(f)\,\partial_\mu f_a^0\,\partial^\mu f_b^0 + B_{ij}^{ab}\,\partial_\mu f_a^0\,\partial^\mu f_b^{(ij)} + C_{ijkl}^{ab}\,\partial_\mu f_a^{(ij)}\,\partial^\mu f_b^{(kl)}\,. \tag{2.26}$$

In the particular case of $n=2$, one has 3 scalar weights type $P(\lambda,\Omega)$ which can be put as a real weight $P(\lambda,\Omega)$ and a complex $Q(\lambda,\Omega)$. The model (2.5) reads therefore in the complex representations of $\Omega = \Omega^1 + i\Omega^2$ as:

$$\mathcal{L}_\sigma = P(\lambda,\Omega,\bar{\Omega})\left|D^{++}\Omega\right|^2 + \bar{Q}(\lambda,\Omega,\bar{\Omega})(D^{++}\Omega)^2 + Q(\lambda,\Omega,\bar{\Omega})(D^{++}\bar{\Omega})^2 \tag{2.27}$$

If under global phase transformations the scalar weights P and Q transform as:

$$\begin{aligned} P\left(\lambda, e^{i\alpha}\Omega, e^{-i\alpha}\bar{\Omega}\right) &= P(\lambda,\Omega,\bar{\Omega}) \\ \bar{Q}\left(\lambda, e^{i\alpha}\Omega, e^{-i\alpha}\bar{\Omega}\right) &= e^{2i\alpha}\,\bar{Q}(\lambda,\Omega,\bar{\Omega})\,; \end{aligned} \tag{2.28}$$

the model (2.27) has a $U(1)$ symmetry.

The equation of motion of (2.27) reads as:

$$D^{++2}\Omega + \frac{P_{,\Omega}}{P}(D^{++}\Omega)^2 + \frac{2Q}{P}\left[D^{++2}\bar{\Omega} + \frac{Q_{,\bar{\Omega}}(D^{++}\bar{\Omega})^2 + 2Q_{,\Omega}\left|D^{++}\Omega\right|^2 - \bar{Q}_{,\bar{\Omega}}(D^{++}\Omega)^2}{2Q}\right] = 0$$

$$Q_{,\Omega} = \frac{\partial Q}{\partial \Omega} \quad , \quad \bar{Q}_{,\bar{\Omega}} = \frac{\partial \bar{Q}}{\partial \bar{\Omega}} \tag{2.29}$$

Similarly as in Sec. 2, Eq.(2.29) carries both dynamical and constraint equations. The analogue of the relation (2.11), which expresses the higher $SU(2)$ auxiliary field representations of Eq.(2.3) in terms of the $N=2$ on-shell degrees of freedom $f^0, f^{(ij)}$ and their conjugation, reads as:

$$\partial^{++2}\omega + \frac{P_{,\omega}}{P}(\partial^{++}\omega)^2 + \frac{2Q}{P}\Big[\partial^{++2}\bar{\omega} + \frac{1}{2Q}\left(Q_{,\bar{\omega}}(\partial^{++}\bar{\omega})^2 + 2Q_{,\omega}\left|\partial^{++}\omega\right|^2 - \bar{Q}_{,\bar{\omega}}(\partial^{++}\omega)^2\right)\Big] = 0 \tag{2.30}$$

Eq.(2.30) is a non-linear differential equation of second order on the sphere S^2 depending on both ω and $\bar{\omega}$. Such an equation is difficult to solve in general. The difficulty is due to: 1) it is non-linear in $\partial^{++}\omega$ and $\partial^{++}\bar{\omega}$; 2) The coefficients are non constant and finally it is non homogeneous since it depends both on ω and $\bar{\omega}$. To overcome these difficulties, one needs an extra symmetry.

In the next section, we shall discuss the particular case of the Taub-Nut metric with a global $U(1)\times SU(2)$ symmetry. Before that let us give the component field form of the Lagrangian (2.7):

$$\begin{aligned} \mathcal{L}_\sigma = {} & A(f,\bar{f})\left|\partial_\mu f^0\right|^2 + \bar{B}(f,\bar{f})(\partial_\mu f^0)^2 + B(f,\bar{f})(\partial_\mu \bar{f}^0)^2 \\ & + \bar{C}_{ij}\,\partial_\mu f^0 \partial^\mu f^{(ij)} + C^{ij}\,\partial_\mu \bar{f}^0 \partial^\mu \bar{f}_{(ij)} \\ & + \bar{d}_{ij}\,\partial_\mu \bar{f}^0 \partial^\mu f^{(ij)} + d^{ij}\,\partial_\mu f^0 \partial^\mu \bar{f}_{(ij)} \\ & + \bar{e}_{ijkl}\,\partial_\mu f^{(ij)} \partial^\mu f^{(kl)} + e^{ijkl}\,\partial_\mu \bar{f}_{(ij)} \partial^\mu \bar{f}_{(kl)} \\ & + h_{ij}{}^{kl}\,\partial_\mu f^{(ij)} \partial^\mu \bar{f}_{(kl)}\,. \end{aligned} \tag{2.31}$$

3. THE TAUB-NUT METRIC IN THE Ω REPRESENTATION

In this section, we study the case of the Taub-Nut model in the Ω representation. The Taub-Nut metric [7] has a $U(2) \simeq U(1)\times SU(2)$ isometry group and it is shown to be hyperkahler. The $N=2$ supersymmetric non linear σ-model (2.31) constrained by a global $U(1)$ symmetry may describe the Taub-Nut metric in the Ω representation. Before deriving the appropriate Taub-Nut model and working out the corresponding metric as an illustrating example of Eq.(2.31), we would like to note that a dual form of this model in terms of the Fayet-Sohnius hypermultiplet [19](referred to as ϕ^+ representation here below) has been discussed in Refs. [11],[13],[14].

Now, we come to derive the Taub-Nut model in the Ω-representation. As mentioned earlier this model has a $U(2)$ symmetry and can therefore be obtained by coupling a complex HST hypermultiplet to a $U(1)$ auxiliary vector superfield V^{++}:

$$L = \int d^4\theta^+ \left\{ \left|(D^{++} - iV^{++})\Omega\right|^2 + \frac{1}{2\lambda} V^{++2} \right\} \tag{3.1}$$

$[\lambda] = -2$, a dimensionfull coupling parameter.

The equations of motion of Ω and V^{++} read as:

$$(D^{++} - iV^{++})^2\Omega = 0 \tag{3.2}$$

$$V^{++} = \overset{*}{V}{}^{++} = -i\lambda \frac{\overset{*}{\Omega}\, \overleftrightarrow{D}{}^{++}\Omega}{(1 + 2\lambda\,\Omega\overset{*}{\Omega})} \tag{3.3}$$

Putting Eq.(3.3) into (3.1), one gets a Lagrangian of the form (2.27), where $P(\lambda,\Omega,\bar\Omega)$ and $Q(\lambda,\Omega,\bar\Omega)$ are given by:

$$P(\lambda,\Omega,\bar\Omega) = 1 - \frac{\lambda\,\Omega\,\overset{*}{\bar\Omega}}{(1+2\lambda\Omega\overset{*}{\bar\Omega})} \tag{3.4}$$

$$\bar Q(\lambda,\Omega,\bar\Omega) = \frac{\lambda\,\overset{*}{\bar\Omega}{}^2}{2(1+2\lambda\Omega\overset{*}{\bar\Omega})} \quad . \tag{3.5}$$

These relations satisfy the property (2.28) and lead to a $U(1)$ invariant Lagrangian.

Now we want to determine the metric tensors of the $N=2$ non-linear σ-model (2.31) associated with the Taub-Nut metrics(3.4) and (3.5). To that purpose we have to solve the constraint equation obtained by projecting out the auxiliary field equations from the relation (3.2) namely:

$$\left[D^{++} + \lambda\frac{\Omega\,\overleftrightarrow{D}^{++}\overset{*}{\bar\Omega}}{(1+2\lambda\Omega\bar\Omega)}\right]^2\Omega = 0 \tag{3.6}$$

Eq.(3.6) is a non-linear differential equation on the sphere S^2. Notice however that this equation may be linearized by remarking that

$$D^{++}V^{++} = D^{++}\left[\frac{\overset{*}{\bar\Omega}\,\overleftrightarrow{D}^{++}\Omega}{(1+2\lambda\Omega\overset{*}{\bar\Omega})}\right] = 0 \quad . \tag{3.7}$$

Therefore the quantity V^{++} (3.3) is just the Noether current associated with the global $U(1)$ symmetry (2.28).

Moreover, since Eq.(3.6) involves a second order harmonic differential operator, one can introduce another auxiliary superfield (as Lagrange parameter) to render the constraint equation as first order harmonic differential equations easy to solve. Thus, the appropriate linearized action associated to Eq.(3.1) contains a complex auxiliary superfield V^{++} and reads as:

$$\mathcal{L} = \int d^4\theta^+\left\{V^{++}D^{++}\overset{*}{\bar\Omega} - \frac{1}{2}V^{++}\overset{*}{\bar V}{}^{++} - \lambda\left[\Omega^2\overset{*}{\bar V}{}^{++2} - |\Omega V^{++}|^2\right] + h.c.\right\} \tag{3.8}$$

V^{++} has now a c.d equal to 1.

The above form of the Lagrangian depends linearly on the harmonic derivative and leads to coupled equations of motion

$$D^{++}\Omega + 2\lambda N^{++}\Omega + V^{++} = 0 \tag{3.9}$$

$$D^{++}V^{++} + 2\lambda N^{++}V^{++} = 0 \tag{3.10}$$

$$N^{++} = V^{++}\overset{*}{\bar\Omega} - \overset{*}{\bar V}{}^{++}\Omega = -\overset{*}{\bar N}{}^{++} \tag{3.11}$$

The solving of the second order differential equations (3.6) is then equivalent to solving two coupled linear differential equations (3.9) and (3.10).

Similarly as for Eq.(3.7), Eqs.(3.9) and (3.10) are in fact linear and therefore solvable since we have:

$$D^{++}N^{++} = 0 \tag{3.12}$$

This conservation law is also associated with the global $U(1)$ symmetry of the Lagrangian (3.8)

$$\Omega' = e^{i\alpha}\Omega$$
$$V^{++\prime} = e^{i\alpha}V^{++} \tag{3.13}$$

Furthermore, it is interesting to note the two following points:

1) Making Ω and V^{++} real, the interacting part of Eq.(3.8) proportional to λ vanishes and the theory becomes free. This feature may be seen also in Eqs. (3.4) and (3.5).

2) The Lagrangian (3.8) has a dual form using the ϕ^+ representation [15],[16].

Denoting the bosonic component fields of the superfield V^{++} as:

$$V^{++} = i\sigma^{++} + \theta^{+2}M + \bar\theta^{+2}N + i\theta^+\sigma^\mu\bar\theta^+K_\mu + \theta^{+2}\bar\theta^{+2}D^{(--)} , \tag{3.14}$$

the kinematical parts of the superfields Eqs. (3.9) and (3.10) take the form:

$$\begin{cases} \partial^{++}\omega + 2i\lambda\left(\bar\omega\sigma^{++} + \omega\bar\sigma^{++}\right)\omega + i\bar\sigma^{++} = 0 \\ \partial^{++}\sigma^{++} + 2i\lambda\left(\bar\omega\sigma^{++} + \omega\bar\sigma^{++}\right)\sigma^{++} = 0 \end{cases} \tag{3.15}$$

$$\begin{cases} \partial^{++}A_\mu^{--} - 2\partial_\mu \omega + 2i\lambda(\bar{\omega}\sigma^{++} + \omega\bar{\sigma}^{++})A_\mu^{--} + 2\lambda\omega Z_\mu + K_\mu = 0 \\ \partial^{++}K_\mu - 2i\partial_\mu\sigma^{++} + 2i\lambda(\bar{\omega}\sigma^{++} + \omega\bar{\sigma}^{++})K_\mu + 2i\lambda\sigma^{++}Z_\mu = 0 \end{cases} \tag{3.16}$$

$$\begin{cases} \partial^{++}F^{--} + 2i\lambda(\bar{\omega}\sigma^{++} + \omega\bar{\sigma}^{++})F^{--} + 2\lambda(\bar{\omega}M - \omega\bar{N})\omega + 2i\lambda(\bar{G}^{--}\sigma^{++} + F^{--}\bar{\sigma}^{++})\omega = 0 \\ \partial^{++}M + 2i\lambda(\bar{\omega}\sigma^{++} + \omega\bar{\sigma}^{++})M + 2i\lambda(\bar{\omega}M - \omega\bar{N})\sigma^{++} - 2\lambda(\bar{G}^{--}\sigma^{++} + F^{--}\bar{\sigma}^{++})\sigma^{++} = 0 \end{cases} \tag{3.17}$$

$$\begin{cases} \partial^{++}G^{--} + 2i\lambda(\bar{\omega}\sigma^{++} + \omega\bar{\sigma}^{++})G^{--} + 2\lambda(\bar{\omega}N - \omega\bar{M})\omega + 2i\lambda(\bar{F}^{--}\sigma^{++} + G^{--}\bar{\sigma}^{++})\omega = 0 \\ \partial^{++}N + 2i\lambda(\bar{\omega}\sigma^{++} + \omega\bar{\sigma}^{++})N + 2i\lambda(\bar{\omega}N - \omega\bar{M})\sigma^{++} - 2\lambda(\bar{F}^{--}\sigma^{++} + G^{--}\bar{\sigma}^{++})\sigma^{++} = 0\,, \end{cases} \tag{3.18}$$

where

$$Z_\mu = \bar{\omega}K_\mu - \omega\bar{K}_\mu + A_\mu^{--}\bar{\sigma}^{++} - \bar{A}_\mu^{--}\sigma^{++} \tag{3.19}$$

To evaluate the solutions of these equations, we take advantage of the U(1) symmetry (3.13) to choose the form of the solutions as:

$$\omega(f) = f\ \exp(-i\lambda\beta) \tag{3.20}$$

$$\sigma^{++}(f) = \gamma^{++}(f)\ \exp(-i\lambda\beta) \tag{3.21}$$

$$A_\mu^{--}(f) = B_\mu^{--}(f)\ \exp(-i\lambda\beta) \tag{3.22}$$

$$K_\mu(f) = V_\mu(f)\ \exp(-i\lambda\beta) \tag{3.23}$$

etc...

Putting Eqs (3.20)-(3.23) into Eqs.(3.15) and (3.16), one gets:

$$\partial^{++}\beta = 2(\bar{\omega}\sigma^{++} + \omega\bar{\sigma}^{++}) \tag{3.24}$$

$$\gamma^{++} = i f^{++} = i f^{(ij)} U_i^+ U_j^+ \tag{3.25}$$

$$\partial^{++}B_\mu^{--} - 2\partial_\mu f + 4\lambda f Z_\mu + V_\mu = 0 \tag{3.26}$$

$$\partial^{++}V_\mu + 2\partial_\mu f^{++} - 4\lambda f^{++} Z_\mu = 0 \tag{3.27}$$

and similar relations for Eqs.(3.17) and (3.18).

Integrating relation (3.8) with respect to θ^+ and $\bar{\theta}^+$ and using the constraint Eqs. (3.15)-(3.18) one obtains the simple form

$$\mathcal{L}_\sigma^{TN} = -\frac{1}{2}\left\{\left(\bar{f}\,\partial_\mu V^\mu + \partial^{++}B_\mu^{--}\,\partial^\mu \bar{f} - \frac{\lambda}{2} Z_\mu Z^\mu\right) + hc\right\} \tag{3.28}$$

To evaluate the above Lagrangian, it remains to determine the quantities β, V_μ and Z_μ. From Eq.(3.24) and (3.25), one obtains for β:

$$\begin{aligned} \beta(f) &= 2(f^0\bar{f} - \bar{f}^0 f) + (f^{--}\bar{f}^{++} - f^{++}\bar{f}^{--}) + S \\ &= \left[f^0\bar{f}^{(ij)} - \bar{f}^0 f^{(ij)} + \varepsilon_{lk} f^{(ik)}\bar{f}^{(jl)}\right] U^+_{(i} U^-_{j)} + S\,, \end{aligned} \tag{3.29}$$

where the extra piece S represents an SU(2) singlet quantity resulting from the integration process over the harmonic variables [13]. As examples, S may be chosen as

$$S_0 \sim 1/\lambda \tag{3.30}$$

$$S_1 \sim |f^0|^2 + \frac{1}{6} f^i_{\ j}\, \bar{f}^{(j}_{\ \ i)} \tag{3.31}$$

or more generally any function depending on singlet quantities, such as (3.31), respecting the c∙d two. Notice that β consists of an SU(2) triplet quantity and the arbitrariness is carried by the singlet part. In what follows we choose the fixing condition (3.30) or equivalently up to a phase transformation (3.13) S is equal to zero.

The solution of V_μ reads after some algebra in the following form:

$$V_\mu = -2\partial_\mu f + 4\lambda f Z_\mu - 2\lambda H_\mu \tag{3.32}$$

where

$$H_\mu = 2 f^{--}\partial_\mu\left[f\bar{f}^{++} - \bar{f} f^{++}\right] \tag{3.33}$$

Or equivalently by writing

$$H_\mu = H_\mu^0 + H^{(ij)} U_i^+ U_j^- + H^{(ijkl)} U_i^+ U_j^+ U_k^- U_l^- \tag{3.34}$$

with

$$\begin{aligned} H_\mu^0 = -\frac{1}{6} \varepsilon_{i(k} \varepsilon_{l)j} \Big\{ & f^{(ij)} \partial_\mu \left[f^0 \bar{f}^{(kl)} - \bar{f}^0 f^{(kl)} \right] \\ & + \varepsilon_{mn} f^{(ij)} \partial_\mu \left[\bar{f}^{(km)} f^{(nl)} \right] \Big\} \end{aligned} \tag{3.35}$$

$$\begin{aligned} H^{+-} = -\frac{1}{12} f^{(ij)} \partial_\mu & \left[f^0 \bar{f}^{(kl)} - \bar{f}^0 f^{(kl)} + \varepsilon_{mn} \bar{f}^{(km)} f^{(ln)} \right] \times \\ & \left[\varepsilon_{jk} U_{(i}^+ U_{l)}^- + \varepsilon_{ik} U_{(j}^+ U_{l)}^- + \varepsilon_{jl} U_{(i}^+ U_{k)}^- + \varepsilon_{il} U_{(j}^+ U_{k)}^- \right] \end{aligned} \tag{3.3(}$$

and

$$H^{++--} = \frac{1}{12} f^{(ij)} \partial_\mu \left[f^0 \bar{f}^{kl} - \bar{f}^0 f^{(kl)} + \varepsilon_{mn} \bar{f}^{(km)} f^{(nl)} \right] U_{(i}^+ U_j^+ U_k^- U_{l)}^- \,. \tag{3.37}$$

Finally, the determination of Z_μ, which is the most tedious part, may be obtained as follows:

Write Z_μ as

$$Z_\mu = Z_\mu^0 + Z_\mu^{+-} \tag{3.38}$$

where

$$\partial^{++} Z_\mu = Z_\mu^{++} = \partial^{++} \partial_\mu \beta \tag{3.39}$$

and Z_μ^0 is the SU(2) singlet part of Z_μ:

$$Z_\mu^0 = \int dU \, Z_\mu = \int dU \left(\bar{f} v_\mu + i \bar{f}^{++} B_\mu^{--} - hc \right) . \tag{3.40}$$

Integrating Eq. (3.39), one obtains, using Eq.(3.29)

$$Z_\mu^{+-} = \partial_\mu \beta + SU(2) \text{ singlet chosen zero,} \tag{3.41}$$

and from Eqs. (3.26), (3.32) and (3.40) one gets after long calculations:

$$\begin{aligned} Z_\mu^0 = \frac{1}{1-\lambda\rho^2} \Big\{ & -2 \bar{f}^0 \overleftrightarrow{\partial}_\mu f^0 + \bar{f}_{ij} \overleftrightarrow{\partial}_\mu f^{(ij)} \\ & - \frac{4\lambda}{3} \left(\bar{f}_0 f_{(ij)} - f_0 \bar{f}_{(ij)} \right) \partial_\mu \left[f_0 \bar{f}_{(ij)} - \bar{f}_0 f_{(ij)} + \varepsilon_{mn} \bar{f}^{(im)} f^{(jn)} \right] \Big\} , \end{aligned} \tag{3.42}$$

with

$$\rho^2 = 8 \left[f^0 \bar{f}^0 + f^{(i}{}_{j)} f^j{}_i \right] \tag{3.43}$$

Now, putting Eqs.(3.29)-(3.42) into the Lagrangian (3.28), one gets a quantity depending on f^0, $f_{(ij)}$, their conjugates and at most eight harmonic variables. The integration with respect to these variables gives the pure bosonic part of the N=2 supersymmetric Taub-Nut model whose metric tensors in the Ω representation (2.31) are given by:

$$\begin{aligned} A(f) = {} & 4(3 - \lambda f\bar{f}) + \frac{8\lambda}{(1-\lambda\rho^2)} \left\{ \frac{2\lambda}{3} \left(f_0^2 \bar{f}^2 + \bar{f}_0^2 f^2 \right) \left(4 + \frac{38}{3} f\bar{f} \right) \right\} \\ & + \frac{8\lambda}{(1-\lambda\rho^2)} \Big\{ |f_0|^2 \Big[7 - 8\lambda |f_0|^2 (6 + 4\lambda f\bar{f}) + \\ & + \frac{4\lambda}{3} \left(22 f\bar{f} + \frac{37}{3} \lambda (f\bar{f})^2 + \frac{\lambda}{3} f^2 \bar{f}^2 - 12\lambda f_0^2 \bar{f}^2 - 12\lambda \bar{f}_0^2 f^2 \right) \Big] \Big\} , \end{aligned} \tag{3.44}$$

$$\begin{aligned} \bar{B}(f) = {} & -4\lambda \left(f_0^2 - \bar{f}_0^2 \right) + \frac{2}{3} \lambda \left(f^2 + 2 \bar{f}^2 \right) \\ & - \frac{\lambda}{(1-\lambda\rho^2)^2} \left\{ 28 |f_0|^2 + \frac{8\lambda}{3} \bar{f}_0^2 \left(44 + \frac{74}{3} \lambda f\bar{f} \right) f\bar{f} + \frac{16}{9} \lambda^2 f_0^2 \bar{f}^4 \right\} \\ & + \frac{8\lambda |f_0|^2}{(1-\lambda\rho^2)^2} \left\{ 8\lambda \bar{f}_0^2 (3 + 2\lambda f\bar{f}) - \frac{2\lambda}{3} \bar{f}^2 \left(4 + \frac{38}{3} \lambda f\bar{f} \right) + 16\lambda^2 |f_0|^2 \bar{f}^2 \right\} , \end{aligned} \tag{3.45}$$

$$
\begin{aligned}
\bar{C}_{ij}(f) = {} & \frac{16}{3}\lambda \bar{f}_0 \bar{f}_{(ij)} + \frac{2\lambda}{3}\varepsilon^{mn}\bar{f}_{(in)}\bar{f}_{(jm)} \\
& + \frac{8\lambda}{(1-\lambda\rho^2)^2}\Big\{ \bar{f}_{(ij)}\Big[2\bar{f}_0 + \frac{\lambda}{3} f_0 \bar{f}^2 + \frac{19}{3}\bar{f}_0 (f\bar{f}) + {} \\
& \qquad \lambda|f_0|^2\Big(-\frac{28}{3}\bar{f}_0 - 16\lambda \bar{f}_0|f_0|^2 + \frac{4}{9}\lambda f_0 \bar{f}^2 + \frac{76}{9}\lambda \bar{f}_0 f\bar{f}\Big)\Big] \\
& \qquad + \lambda \bar{f}_0^2 f_{(ij)}\Big[\bar{f}_0\Big(\frac{8}{3} + \frac{76}{9}\lambda f\bar{f} - 16\lambda|f_0|^2\Big) + \frac{4}{9}\lambda f_0 f^2\Big]\Big\} \\
& + \frac{8\lambda\, \bar{f}_{(in)} f_{(jm)}\varepsilon^{mn}}{(1-\lambda\rho^2)^2}\Big[\lambda \bar{f}_0^2\Big(\frac{4}{3}+\frac{68}{9}\lambda f\bar{f} - 16\lambda|f_0|^2\Big) + \frac{4}{9}\lambda|f_0|^2\lambda\bar{f}^2\Big] \\
& + \frac{8\lambda\, \bar{f}_{(in)}\bar{f}_{(jm)}\varepsilon^{mn}}{(1-\lambda\rho^2)^2}\Big[\frac{4}{9}\lambda^2 f_0^2\bar{f}^2 + {} \\
& \qquad \lambda|f_0|^2\Big(\frac{8}{3}+\frac{68}{9}\lambda f\bar{f} + 16\lambda|f_0|^2\Big)\Big]
\end{aligned}
\tag{3.46}
$$

$$
\begin{aligned}
\bar{d}_{ij} = {} & -4\lambda f_0\bar{f}_{(ij)} - \frac{4\lambda}{3}\bar{f}_0 f_{(ij)} + \frac{\lambda}{3}\varepsilon^{mn}\Big[f_{(im)}\bar{f}_{(jn)} + \bar{f}_{(im)} f_{(jm)}\Big] \\
& - \frac{8\lambda}{(1-\lambda\rho^2)^2}\Big\{\bar{f}_{(ij)}\Big[2f_0 + \frac{19}{3}f_0 f\bar{f} + \frac{\lambda}{3}\bar{f}_0 f^2\Big] \\
& \qquad - \lambda|f_0|^2\Big[\frac{28}{3}f_0 + 16\lambda f_0|f_0|^2 - \frac{76}{9}\lambda f_0 f\bar{f} \\
& \qquad\qquad - \frac{4}{9}\lambda\bar{f}_0 f^2\Big] \\
& \qquad + f_{(ij)}\Big[\frac{4}{9}\lambda^2\bar{f}_0^3 f^2 + \lambda|f_0|^2\Big(\frac{8}{3}\bar{f}_0 - 16\lambda|f_0|^2\bar{f}_0 + \frac{76}{9}\lambda\bar{f}_0 f\bar{f}\Big)\Big]\Big\} \\
& - \varepsilon^{mn}\frac{8\lambda}{(1-\lambda\rho^2)^2}\bar{f}_{(in)} f_{(jm)}\Big[\frac{4}{9}\lambda^2\bar{f}_0^2 f^2 + \lambda|f_0|^2\Big(\frac{8}{3} - 16\lambda|f_0|^2 + \frac{68}{9}\lambda f\bar{f}\Big)\Big] \\
& - \varepsilon^{mn}\frac{8\lambda}{(1-\lambda\rho^2)^2}\bar{f}_{(in)}\bar{f}_{(jm)}\Big[\lambda f_0^2\Big(\frac{8}{3}+\frac{68}{9}\lambda(f\bar{f})\Big) + \lambda|f_0|^2\Big(\frac{4}{9}\lambda f^2 - 16\lambda\bar{f}_0^2\Big)\Big]
\end{aligned}
\tag{3.47}
$$

$$
\begin{aligned}
\bar{e}_{ijkl} = {} & \frac{\lambda}{3}\Big[6f_0^2 - \bar{f}_0^2\Big]\varepsilon_{i(k}\varepsilon_{l)j} - \frac{\lambda}{10}\bar{f}^2\varepsilon_{i(k}\varepsilon_{l)j} \\
& - \frac{8\lambda}{3}\varepsilon_{il}\bar{f}_0\bar{f}_{(jk)} - \frac{\lambda}{5}\bar{f}_{(ij)}\bar{f}_{(kl)} - \frac{12}{15}\lambda\bar{f}_{(ik)}\bar{f}_{(jl)} \\
& - \frac{6}{15}\varepsilon^{mm}\varepsilon_{jk}\bar{f}_{(im)}\bar{f}_{(lm)} + (i\longleftrightarrow j) \\
& \frac{-4\lambda}{(1-\lambda\rho^2)^2}\Big\{\bar{f}_{(ij)}\bar{f}_{(kl)} + \frac{8}{3}\lambda\Big[\bar{f}_0^2 f_{(kl)} + |f_0|^2\bar{f}_{(kl)} + {} \\
& \qquad + \bar{f}_0 f_{(ml)}\bar{f}_{(km)}\varepsilon^{mm} + f_0\bar{f}_{(ml)}\bar{f}_{(km)}\varepsilon^{mm}\Big]\bar{f}_{(ij)} \\
& \qquad + \frac{4\lambda^2}{9}\Big[\bar{f}_0^2\Big(\bar{f}_0 f_{(ij)} + f_0\bar{f}_{(ij)}\Big)\Big(\bar{f}_0 f_{(kl)} + f_0\bar{f}_{(kl)}\Big) \\
& \qquad + \varepsilon^{mn}\varepsilon^{np}\Big(\bar{f}_0\bar{f}_{(in)} f_{(jm)} + f_0\bar{f}_{(in)}\bar{f}_{(jm)}\Big)\Big(\bar{f}_0\bar{f}_{(kn)} f_{(lp)} + f_0\bar{f}_{(kn)}\bar{f}_{(lp)}\Big) \\
& \qquad + 2\Big(\bar{f}_0^2 f_{(ij)} + |f_0|^2\bar{f}_{(ij)}\Big)\Big(\bar{f}_0 f_{(ml)} + f_0\bar{f}_{(ml)}\Big)\bar{f}_{(km)}\varepsilon^{mm}\Big]\Big\}
\end{aligned}
\tag{3.48}
$$

and finally

$$
\begin{aligned}
h_{ij}{}^{k\ell} = & -\delta_{(i}{}^{k}\delta_{j)}^{\ell}\left[1-\frac{10}{3}\lambda|f_0|^2+\frac{\lambda}{5}f\bar{f}\right] \\
& +\left[\frac{4\lambda}{3}\left(\bar{f}_0 f_{(i}{}^{k)}+f_0\bar{f}_{(i}{}^{k)}\right)\delta_j^{\ell}+i\longleftrightarrow j\right] \\
& +\frac{\lambda}{5}\left(f_{(ij)}\bar{f}^{(k\ell)}+\bar{f}_{(ij)}f^{(k\ell)}\right)+\left[\frac{2}{5}\lambda\left(f_{(i}{}^{k)}\bar{f}_{(j}{}^{\ell)}+\bar{f}_{(i}{}^{k)}f_{(j}{}^{\ell)}\right)+(i\longleftrightarrow j)\right] \\
& +\left[\frac{2\lambda}{5}\left(f_{(im)}\bar{f}^{(\ell m)}+\bar{f}_{(im)}f^{(\ell m)}\right)\delta_j{}^{k}+(i\longleftrightarrow j)\right] \\
& +\frac{8\lambda}{(1-\lambda\rho^2)^2}\bar{f}_{(ij)}f^{(k\ell)} \\
& +\frac{48\lambda^2}{(1-\lambda\rho^2)^2}\Big[\bar{f}_{(ij)}\left(|f_0|^2f^{(k\ell)}+f_0^2\bar{f}^{(k\ell)}\right)+f^{(k\ell)}\left(|f_0|^2\bar{f}_{(ij)}+\bar{f}_0^2 f_{(ij)}\right) \\
& \qquad +\varepsilon^{\ell m}\left(\bar{f}_0 f^{(km)}+f_0\bar{f}^{(km)}\right)f_{(mn)}\bar{f}_{(ij)} \\
& \qquad +\varepsilon_{im}\left(\bar{f}_0 f_{(jn)}+f_0\bar{f}_{(jm)}\right)\bar{f}^{(mm)}f^{(k\ell)}\Big] \\
& +\frac{32\lambda^3}{(1-\lambda\rho^2)^2}\Big[|f_0|^2\left(\bar{f}_0 f_{(ij)}+f_0\bar{f}_{(ij)}\right)\left(\bar{f}_0 f_{(k\ell)}+f_0\bar{f}_{(k\ell)}\right) \\
& \qquad +\left(\bar{f}_0\bar{f}_{(in)}f_{(jm)}+f_0\bar{f}_{(in)}\bar{f}_{(jm)}\right)\left(\bar{f}_0 f_{(kr)}f_{(s\ell)}+f_0 f_{(kr)}\bar{f}_{(s\ell)}\right)\varepsilon^{mn}\varepsilon^{sr} \\
& \qquad +2\left(|f_0|^2 f_{(ij)}+f_0^2\bar{f}_{(ij)}\right)\left(\bar{f}_0\bar{f}_{(km)}f_{(n\ell)}+f_0\bar{f}_{(km)}\bar{f}_{(n\ell)}\right)\varepsilon^{nm}\Big].
\end{aligned}
\tag{3.49}
$$

In writing the formulas, we have used the notation

$$
f\bar{f} = f^{(ij)}\bar{f}_{(ij)} \quad ; \quad f^2 = f^{(ij)}f_{(ij)} \; .
$$
$$
\bar{f}^2 = \bar{f}_{(ij)}\bar{f}^{(ij)} \; . \tag{3.50}
$$

Making f_0 and $f_{(ij)}$ real, one can check that Eqs. (3.44)-(3.49) reduce to the free case as mentioned earlier.

4. DISCUSSION AND CONCLUSION

From the present study, we have learnt the following: given an N=2 supersymmetric non-linear σ-model type (2.8), (2.25) in terms of the superfield Ω, one writes down the equations of motion and solves the constraint equations type (2.11)-(2.13) and (3.15)-(3.18). Plug in the component field Lagrangian after which one integrates with respect to the harmonic variables and identifies the explicit hyperkahler metric.

Moreover, through the study of Sec. 2, one learns also that the basic equation corresponds to that which expresses the higher SU(2) representation auxiliary fields of ω in terms of the N=2 on-shell degrees of freedom. This constraint equation comes from the elimination of the "auxiliary field" $\Delta^{(-4)}$ of c-d higher than 2 (Lagrange parameter) and leads to a relation between $\omega = \omega(f)$ and the weights $P_{ab}(\omega)$. The other auxiliary field equations relate the fields F^{--} and A_μ^{--} to $\omega(f)$. Indeed, using Eq.(2.19), the constraints Eqs.(2.12)-(2.13) may be written as expressions completely specified in terms of $\omega(f)$ and its derivatives namely:

$$
\partial^{++2}F^{--}-2\frac{\omega_{,ff}}{\omega_{,f}}f^{++}\partial^{++}F^{--}+\left[2\left(\frac{\omega_{,ff}}{\omega_{,f}}\right)^2-\left(\frac{\omega_{,fff}}{\omega_{,f}}\right)\right]f^{++2}F^{--}=0 \;, \tag{4.1}
$$

$$
\partial^{++2}A_\mu^{--}-4\,\omega_{,f}\,\partial_\mu f^{++}-2\frac{\omega_{,ff}}{\omega_{,f}}f^{++}\partial^{++}A_\mu^{--}+
\left[2\left(\frac{\omega_{,ff}}{\omega_{,f}}\right)^2-\left(\frac{\omega_{,fff}}{\omega_{,f}}\right)\right]f^{++2}A_\mu^{--}=0 \tag{4.2}
$$

Furthermore, to work out an explicit hyperkahler metric, we have to specify the weights $P_{ab}(\Omega)$ and follow the procedure outlined earlier and write down the solutions

$$
\begin{aligned}
\omega &= \omega(f) && \text{(a)} \\
F^{--} &= F^{--}(f) && \text{(b)} \qquad (4.3) \\
A_\mu^{--} &= A_\mu^{--}(f, \partial_\mu f) && \text{(c)}
\end{aligned}
$$

For the Taub-Nut case in the 1+3 parametrization, one has:

$$
\begin{aligned}
\omega(f) &= \left(f^0 + f^{(ij)} U_i^+ U_j^- \right) \exp - i\lambda\beta(f) \\
\beta(f) &= \left[f^0 \bar{f}^{(ij)} - \bar{f}^0 f^{(ij)} + \varepsilon_{mn} f^{(im)} \bar{f}^{(jn)} \right] U_{(i}^+ U_{j)}^- \qquad (4.4)
\end{aligned}
$$

etc ...

The solutions (Eqs.(4.3)) however, are not in general easy to find. This is because Eqs.(2.11)-(2.13) are non-linear differential equations on the S^2 sphere difficult to solve. Nevertheless, as long as hyperkahler metrics building is concerned, one may invert the previous reasoning by giving up an ad hoc solution $\omega = \omega(f)$ for instance

$$
\begin{aligned}
\omega(f) &= f(1 + \lambda f^2) \,, \\
\omega_{,f} &= 1 + 3\lambda f^2 \,, \\
\omega_{,ff} &= 6\lambda f \,, \quad \omega_{,fff} = 0 \qquad (4.5)
\end{aligned}
$$

and try to solve the non-linear differential equations (4.1) and (4.2). The solutions of these equations are not obvious. But for particular choices of (4.3a), one might find solutions and obtain explicit hyperkahler metrics. In this procedure, one overpasses the major difficulty of solving the basic constraint equation type (2.11).

To conclude, we have studied the N=2 supersymmetric non-linear σ-models in the HST representation. We have shown that a class of 4n dimensional N=2 non-linear σ-models are in general completely characterized by giving n(n+1)/2 Kahler like potentials. Four dimensional hyperkahler manifolds preserving the SU(2) symmetry are in particular specified by one Kahler like potential and therefore similar to the N=1 case. These models may be used to describe hyperkahler metrics with SU(2)xG isometry group. The SU(2)xSO(2)≅SU(2)xU(1) Taub-Nut metric is an example. Moreover, this construction recovers a wide class of hyperkahler metrics and in particular for four-dimensional manifolds one may build different types of explicit hyperkahler metrics by giving different solutions of Eq.(2.11). This technique is then a powerful method of constructing hyperkahler metrics.

We have discussed in detail the Taub-Nut model as an illustrating example. We have given the procedure of constructing the action and showed how the constraint equations could be linearized. This method is expected to be applied for all hyperkahler models with SU(2)xG isometry group. The different steps of calculations and the final Taub-Nut metric in the HST representation are shown explicitly. Finally, we would like to note that it is interesting to find the duality transformations which transform the Taub-Nut metric in the 1+3 representation to that of 2+2 one [11].

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REFERENCES

[1] A. Perelomov, Phys. Rep. C146 (1987) 135;
A. Morozov and A. Perelomov, Nucl. Phys. B271 (1986) 620;
J. Gates, C. Hull and M. Roček, Nucl. Phys. B248 (1984) 158.

[2] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, (Interscience, New York, 1969);
L. Alvarez Gaume and D.Z. Freedman, Unification of the Fundamental Particle Interactions, Erice, Eds. S. Ferrara, J. Ellis and P. van Nieuwenhuizen (Plenum Press, New York, 1986).
J.N. Helayel, R. Iengo, F. Legovini and S. Pugnetti, SISSA, Trieste, preprint 38/87/EP (1987);
M. Evans and B.A. Ovrut, Phys. Lett. B190 (1987) 63.

[3] E. Calabi, in Algebraic Geometry and Topology, A Symposium in Honour of S. Lefschetz (Princeton University Press, Princeton, New Jersey, 1957).

[4] S.T. Yau, Proc. Nat. Acad. Sci. 74 (1977) 1798.

[5] L. Alvarez-Gaume and D.Z. Freedman, Comm. Math. Phys. 80 (1981) 443.

[6] E. Sokatchev and K. Stelle, Class. Quant. Grav. 4 (1987) 501-508;
T. Lhallabi and E.H. Saidi, to appear in Int. J. Mod. Phys. A (1987);
C. Hull, Nucl. Phys. B260 (1985) 182;
L. Alvarez-Gaume and P. Ginsparg, Comm. Math. Phys. 102 (1985) 311.

[7] T. Eguchi, P. Gilkey and A. Hanson, Phys. Rep. C66 (1980) 213.

[8] C. Lovelace, Phys. Lett. B135 (1984) 75.

[9] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 1 (1984) 469-498.

[10] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 2 (1985) 601-617.

[11] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, Comm. Math. Phys. 103 (1986) 515-526.

[12] A. Galperin, E. Ivanov, V. Ogievetsky and P.K. Townsend, Class. Quant. Grav. 3 (1986) 625-633.

[13] A. El-Hassouni, T. Lhallabi, E.G. Oudrhiri-Safiani and E.H. Saidi, ICTP, Trieste, preprint, IC/87/205 (1987).

[14] D. Olivier and G. Valent, LPTHE preprint 87-27 (1987).

[15] A. Galperin, E. Ivanov and V. Ogievetsky, Nucl. Phys. B282 (1987) 74.

[16] E.H. Saidi, "On hypermultiplet theories", ICTP, Trieste, preprint IC/86/245, revised version, to appear in IJMPA (1987).

[17] B. Zumino, Phys. Lett. B87 (1979) 203.

[18] J. Bagger, SLAC Pub. 3461 (1984).

[19] P. Fayet, Nucl. Phys. B113 (1976) 135; Nucl. Phys. B149 (1979) 257;
M. Sohnius, Nucl. Phys. B138 (1978) 109;
Abdus Salam and J. Strathdee, Nucl. Phys. B97 (1975) 293.